FIAN/TD/15-05

Secondary Fields in $D > 2$ Conformal Theories

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Abstract

We consider the secondary fields in D-dimensional space, $D \geq 3$, generated by the non-abelian current and energy-momentum tensor. These fields appear in the operator product expansions $j^a_\mu(x)\varphi(0)$ and $T_{\mu\nu}(x)\varphi(0)$. The secondary fields underlie the construction proposed herein (see [1,2] for more details) and aimed at the derivation of exact solutions of conformal models in $D \geq 3$. In the case of $D = 2$ this construction leads to the known [5] exactly solvable models based on the infinite-dimensional conformal symmetry. It is shown that for $D \geq 3$ the existence of the secondary fields is governed by the existence of anomalous operator contributions (the scalar fields R_j and R_T of dimensions $d_j = d_T = D-2$) into the operator product expansions $j^a_\mu j^b_\nu$ and $T_{\mu\nu}T_{\rho\sigma}$. The coupling constant between the field R_j and the fundamental field is found. The fields R_j and R_T are shown to beget two infinite sets of secondary tensor fields of canonical dimensions $D-2+s$, where s is the tensor rank. The current and the energy-momentum tensor belong to those families, their Green functions being expressed through the Green functions of the fields R_j and R_T correspondingly. We demonstrate that the Ward identities give rise to the closed set of equations for the Green functions of the fields R_j and R_T.

1 Introduction and Discussion of the Results

The articles [1,2] (and the references therein) discuss the construction enabling one to derive a family of exactly solvable models in conformal quantum field theory in D-dimensional space, $D \geq 2$. This construction is based upon the following feature of conformal theory: each fundamental complex field $\varphi(x)$ may be associated (provided that the certain assumptions are made, see below) with two infinite sets of secondary tensor fields [1,2]:

$$P_s(x), P_s^{(s_1)}(x), P_s^{(s_1 s_2)}(x), \ldots, P_s^{(s_1 \ldots s_k)}(x), \ldots \tag{1.1}$$

of dimensions

$$l_s = d + s, \quad s = 1, 2, \ldots$$

Here d is the dimension of the fundamental field $\varphi(x)$, s is the tensor rank. One of these sets of fields is associated with the conserved current $j_\mu(x)$, while the other is associated with the energy-momentum tensor $T_{\mu\nu}(x)$. The secondary fields (1.1) appear in the operator product expansions:

$$j_\mu(x)\varphi(0) = \sum_s [P_s^j],\ j_\mu(x) P_{s_1}^j(0) = \sum_s [P_s^{j(s_1)}],\ j_\mu P_{s_2}^{j(s_1)}(0) = \sum_s [P_s^{j(s_1 s_2)}] \ldots, \tag{1.2}$$

$$T_{\mu\nu}(x)\varphi(0) = \sum_s [P_s^T],\ T_{\mu\nu}(x) P_s^T(0) = \sum_s [P_s^{T(s_1)}],\ T_{\mu\nu}(x) P_{s_2}^{T(s_1)}(0) = \sum_s [P_s^{T(s_1 s_2)}] \ldots \tag{1.3}$$

and may be found from the Ward identities for the Green functions $\langle j_\mu \varphi \ldots \rangle$ and $\langle T_{\mu\nu} \varphi \ldots \rangle$

Here we use the standard notation $[P]$ (see [1] for example) for the total contribution of the field $P(x)$ and all its derivatives. What is essential that in any D-dimensional conformal theory the Green functions $\langle j_\mu \varphi \ldots \rangle$ and $\langle T_{\mu\nu} \varphi \ldots \rangle$ are uniquely determined [1,2] from the Ward identities provided that the fields j_μ and $T_{\mu\nu}$ are conformally irreducible[1]. In addition to irreducibility no other requirements are necessary, but this requirement drastically restricts the choice of possible models [5], see also [1,2].

The secondary fields (1.1) exhibit anomalous transformation properties with respect to internal and conformal symmetry groups and satisfy the anomalous Ward identities [1,2].

Each exactly solvable model is intrinsically related to a special tensor field $Q_s^j(x)$ or $Q_s^T(x)$, $s = 1, 2, \ldots$ of the scale dimension $d+s$. This field is a superposition of secondary fields

$$Q_s(x) = \sum_{s_1 \ldots s_k} \alpha_{s_1 \ldots s_k} P_s^{(s_1 \ldots s_k)}(x), \quad s_1 \leq s_2 \leq \ldots \leq s_k, \quad s_1 + s_2 + \ldots + s_k \leq s - 1,$$

where $\alpha_{s_1 \ldots s_k}$ are the scale factors. These factors are defined by the condition that the anomalous terms in Ward identities for the Green functions $\langle j_\mu Q_s^j \varphi \ldots \rangle$ or $\langle T_{\mu\nu} Q_s^T \varphi \ldots \rangle$ should vanish. This is equivalent to a set of requirements [1,2]

$$\langle j_\mu Q_s^j \varphi \rangle = \langle j_\mu Q_s^j P_{s'}^j \rangle = 0 \quad \text{for all } s', \tag{1.4}$$

[1]The conventional conformal transformations ($j_\mu(x) \xrightarrow{R} (x^2)^{-D+1} g_{\mu\nu}(x) j_\nu(Rx)$ where $Rx = x_\mu/x^2$ and analogously for the energy-momentum tensor $T_{\mu\nu}$) in $D > 2$ give rise to irreducible representations of the conformal group, see [3,4]. The requirements for $j_\mu, T_{\mu\nu}$ and their Green functions which single out the irreducible representations are discussed in [1,2] and, in more detail, in [6].

and analogously for the the energy-momentum tensor $T_{\mu\nu}$ and the field Q^T_s.

This being the case, the fields Q^j_s or Q^T_s have normal transformation properties. Each model is fixed by the requirement that the states $Q^j_s(x)|0\rangle$ or $Q^T_s(x)|0\rangle$ vanish:

$$Q^j_s(x) = 0 \text{ or } Q^T_s(x) = 0.$$

The dimension d of the fundamental field $\varphi(x)$, as well as the coefficients $\alpha_{s_1 \ldots s_k}$, are calculated from the equations (1.4).

Let us briefly discuss this construction for $D = 2$. Notice that for $D \geq 3$ the conformal group is $\frac{1}{2}(D+1)(D+2)$-parametric so we do not employ any analogs of the Virasoro algebra. Nevertheless, if one formally sets $D = 2$ in the above construction, one may obtain [1,2] the well known [6] family of two dimensional conformal models. Traditionally, these models are derived using the different construction [6,7] based on the Virasoro algebra. However, as described in [1], the exact solution of such two dimensional models may be obtained without the Virasoro algebra (i.e., using only the 6-parametric conformal group) in the framework of the approach proposed herein. In particular, in $D = 2$ the fields P^T_s introduced above coincide with the covariant superpositions of the secondary fields $a_{-k_1} \ldots a_{-k_m}\varphi(x)$, where $a_{-k}, k > 0$ are the generators of the Virasoro algebra. In $D = 2$ the states $Q_s(s)|0\rangle$ coincide with the null vectors of the two dimensional conformal algebra, and the requirements (1.4) result in the D-dimensional analog of the Katz formula [8].

In the general case, the fields (1.1) for $D \geq 3$ may exist only if the operator product expansions $j_\mu j_\nu$ and $T_{\mu\nu}T_{\rho\sigma}$ include the anomalous terms [1,2]

$$j_\mu(x)j_\nu(0) = [R_j] + \ldots, \quad T_{\mu\nu}(x)T_{\rho\sigma}(0) = [R_T] + \ldots, \tag{1.5}$$

where R_j and R_T are the scalar fields of dimensions

$$d_{R_j} = d_{R_T} = D - 2.$$

In $D = 2$, the scalar fields R_j and R_T are reduced to constants which coincide with the central charges of the two dimensional theory.

One of the most important goals of the approach proposed in [1,2] for $D \geq 3$ is to establish the equations for the fields R_j and R_T, to find the conditions when the fields (1.1) do exist, and are well defined, and to calculate their Green functions. These problems are, in principle, resolved in this article.

Below we demonstrate that the fields (1.1) exist and have well-defined Green functions only for selected values of normalization constants of the Green functions $\langle\varphi\varphi^+ R_j\rangle$ and $\langle\varphi\varphi R_T\rangle$.

For simplicity we calculate the constant for the function $\langle\varphi\varphi^+ R_j\rangle$ in the non-abelian case[2]. The internal symmetry group generators t^a are chosen to be anti-hermitian

$$(t^a)^+ = -t^a, \; [t^a, t^b] = f^{abc}t^c, \; t^a t^a = -C_g, \; f^{abc}t^b t^c = C_v t^a$$
$$(t^a\varphi)_\alpha = (t^a)^\beta_\alpha \varphi_\beta, \; (t^a\varphi^+)^\alpha = -(\varphi^+)^\beta (t^a)^\alpha_\beta.$$

The field $\varphi(x) = \varphi_\alpha(x)$ will be considered as a scalar field.

Analogous calculations for the energy-momentum tensor are described only qualitatively.

[2]The abelian case is exceptional; it demands setting $R_j = 0$ and taking into account the C-number contribution into the operator product expansion $j_\mu(x)j_\nu(0)$, see [1,2].

2 Requirements for Existence of Secondary Fields

Consider the Green functions $\langle P_s^j \varphi^+ j_\mu^a \rangle$. They are calculated from the equation [1,2]

$$\langle P_s^j \varphi^+ j_\mu^a \rangle \equiv \langle P^j_{\mu_1 \dots \mu_s}(x_1) \varphi^+(x_2) j_\mu^a(x_3) \rangle$$
$$= - \operatorname*{res}_{l=d+s} \int dy_1\, dy_2\, B^l_{\mu_1 \dots \mu_s}(x_1 y_1 y_2) t^b \partial_\nu^{y_2} \langle j_\nu^b(y_2) \varphi(y_1) \varphi^+(x_2) j_\mu^a(x_3) \rangle, \quad (2.1)$$

where

$$B^l_{\mu_1 \dots \mu_s}(x_1 x_2 x_3) = \lambda^{x_1}_{\mu_1 \dots \mu_s}(x_2 x_3) \left(\frac{1}{2} x_{12}^2\right)^{-\frac{l-d-s+D}{2}} \left(\frac{1}{2} x_{13}^2\right)^{-\frac{l+d-s-D}{2}} \left(\frac{1}{2} x_{23}^2\right)^{\frac{l+d-s-D}{2}}$$
$$\lambda^{x_1}_{\mu_1 \dots \mu_s}(x_2 x_3) = \lambda^{x_1}_{\mu_1}(x_2 x_3) \dots \lambda^{x_1}_{\mu_s}(x_2 x_3) - \text{traces},$$
$$\lambda^{x_3}_\mu(x_1 x_2) = \frac{(x_{13})_\mu}{x_{13}^2} - \frac{(x_{23})_\mu}{x_{23}^2}, \qquad (x_{ij})_\mu = (x_i)_\mu - (x_j)_\mu. \quad (2.2)$$

To calculate the r.h.s. of Eq.(2.1) one should use the anomalous Ward identity

$$\partial_\nu^x \langle j_\nu^b(x) j_\mu^a(y) \varphi(x_1) \varphi^+(x_2) \rangle = - \sum_{i=1}^{2} \delta(x - x_i) t^b_{x_i} \langle j_\mu^a(y) \varphi(x_1) \varphi^+(x_2) \rangle$$
$$+ f^{bac} \delta(x-y) \langle j_\mu^c(y) \varphi(x_1) \varphi^+(x_2) \rangle + \partial_\mu^x \delta(x-y) \langle R^{ab}(y) \varphi(x_1) \varphi^+(x_2) \rangle, \quad (2.3)$$

where the notation $t^a_{x_i}$ means that the matrix t^a act on the field $\varphi(x_i)$ in accordance with

$$(t^a \varphi)_\alpha = (t^a)_\alpha^\beta \varphi_\beta, \;\; (t^a \varphi^+)^\alpha = -(\varphi^+)^\beta (t^a)^\alpha_\beta.$$

As the result, the expression under the "res" sign takes the form

$$\left(C_g - \frac{1}{2} C_V\right) \int dx_5\, B^l_{\mu_1 \dots \mu_s}(x_1 x_5 x_2) \langle \varphi(x_5) \varphi^+(x_2) j_\mu^a(x_3) \rangle$$
$$+ \frac{1}{2} C_V \int dx_5\, B^l_{\mu_1 \dots \mu_s}(x_1 x_5 x_3) \langle \varphi(x_5) \varphi^+(x_2) j_\mu^a(x_3) \rangle$$
$$+ t^b \int dx_5 \left[\partial_\mu^{x_3} B^l_{\mu_1 \dots \mu_s}(x_1 x_5 x_3)\right] \langle \varphi(x_5) \varphi^+(x_2) R^{ab}(x_3) \rangle. \quad (2.4)$$

Conformal expressions for the functions $\langle \varphi \varphi^+ j_\mu^a \rangle$ and $\langle \varphi \varphi^+ R^{ab} \rangle$ are (see [1,2]):

$$\langle \varphi(x_1) \varphi^+(x_2) j_\mu^a(x_3) \rangle = g_j t^a \lambda_\mu^{x_3}(x_1 x_2) \Delta_j(x_1 x_2 x_3),$$
$$t^b \langle \varphi(x_1) \varphi^+(x_2) R^{ab}(x_3) \rangle = g_R t^a \Delta_j(x_1 x_2 x_3), \quad (2.5)$$

where g_j and g_R are the normalization constants, and

$$\Delta_j(x_1 x_2 x_3) = \left(\frac{1}{2} x_{12}^2\right)^{-\frac{2d-D+2}{2}} \left(\frac{1}{2} x_{13}^2\right)^{-\frac{D-2}{2}} \left(\frac{1}{2} x_{23}^2\right)^{-\frac{D-2}{2}}.$$

The constant g_j is calculated from the Ward identity for the function $\langle \varphi \varphi^+ j_\mu^a \rangle$; it equals to $g_j = (2\pi)^{-D/2} \Gamma(D/2)$.

Eventually, for the Green function $\langle P_s^j \varphi^+ j_\mu^a \rangle$ in the l.h.s. of Eq. (2.1) we must derive a conformally invariant expression which depends on the normalization constants g_j, g_R. A general expression for the Green function $\langle P_s^j \varphi^+ j_\mu^a \rangle$ is [1]

$$\langle P_s \varphi^+ j_\mu^a \rangle = \lim_{l=d+s} \left[A_1(l,s) C^l_{1\mu, \mu_1 \dots \mu_s}(x_1 x_2 x_3) + A_2(l,s) C^l_{2\mu, \mu_1 \dots \mu_s}(x_1 x_2 x_3)\right], \quad (2.6)$$

where

$$
\begin{aligned}
&C^l_{1\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \lambda^{x_3}_\mu(x_1x_2)\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)\Delta^l_j(x_1x_2x_3),\\
&C^l_{2\mu,\mu_1\ldots\mu_s}(x_1x_2x_3) = \frac{1}{x^2_{13}}\left[\sum_{k=1}^{s} g_{\mu\mu_k}(x_{13})\lambda^{x_1}_{\mu_1\ldots\hat{\mu}_k\ldots\mu_s}(x_3x_2) - \text{traces } \mu_1\ldots\mu_s\right]\Delta^l_j(x_1x_2x_3),\\
&g_{\mu\nu} = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}, \quad \hat{\mu}_k \quad \text{means the omission of the index,}\\
&\Delta^l_j(x_1x_2x_3) = \left(\frac{1}{2}x^2_{12}\right)^{-\frac{l+d-s-D+2}{2}}\left(\frac{1}{2}x^2_{13}\right)^{-\frac{l-d-s+D-2}{2}}\left(\frac{1}{2}x^2_{23}\right)^{-\frac{-l+d+s+D-2}{2}},
\end{aligned}
$$

and the coefficients $A_i(l,s), i = 1,2,$ should be calculated from Eq. (2.4) and depend on the constants g_R, g_j.

It is essential that both terms in Eq. (2.6) are poorly defined in the limit $l = d+2$ because of the factor $\left(\frac{1}{2}x^2_{13}\right)^{-\frac{l-d-s+D}{2}}$ which appears in the contractions over μ, μ_k, $k = 1,\ldots,s$. As shown in [1], for $l = d+s$ their residues have the form

$$
\begin{aligned}
(D-2)\operatorname*{res}_{l=d+s} C^l_{1\mu,\mu_1,\ldots\mu_s}(x_1x_2x_3) &= -\operatorname*{res}_{l=d+s} C^l_{2\mu,\mu_1,\ldots\mu_s}(x_1x_2x_3)\\
&\sim \sum_{k=1}^{s}\Big[(-2)^{-k}\frac{1}{k!}\delta_{\mu\mu_1}\partial^{x_3}_{\mu_2}\ldots\partial^{x_3}_{\mu_k}\delta(x_{13})\frac{(x_{12})_{\mu_{k+1}}}{x^2_{12}}\ldots\frac{(x_{12})_{\mu_s}}{x^2_{12}}\\
&\qquad + \{\ldots\} - \text{traces } \mu_1\ldots\mu_s\Big](x^2_{12})^{-d},
\end{aligned}
\tag{2.7}
$$

where $\{\ldots\}$ are the symmetrizing permutations of the indices $\mu_1\ldots\mu_s$. Hence the non-integrable singularities in Eq. (2.6) cancel if the following condition is satisfied:

$$
\lim_{l=d+s}\frac{A_1(l,s)}{A_2(l,s)} = D-2. \tag{2.8}
$$

In this case, the Green function $\langle P_s\varphi^+ j^a_\mu\rangle$ takes the form [1]

$$
\langle P_s(x_1)\varphi^+(x_2)j^a_\mu(x_3)\rangle \sim t^a(x^2_{23})^{-\frac{D-2}{2}}\overleftrightarrow{\partial}^{x_3}_\mu\left[(x^2_{13})^{-\frac{D-2}{2}}\lambda^{x_1}_{\mu_1\ldots\mu_s}(x_3x_2)\right](x^2_{12})^{-\frac{2d-D+2}{2}} + [\text{q.t.}],
\tag{2.9}
$$

where $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$ and the quasi-local terms [q.t.] are given by the expression (2.7). In what follows, the condition (2.8) will be considered as the requirement for existence of the fields P_s.

Let us substitute (2.5) into (2.4). The first term was calculated in [1] and was shown to satisfy the criterion (2.8). The sum of the second and third terms is calculated using the integral relations listed in [1]. The result is

$$
\begin{aligned}
&t^a\frac{1}{2}g_j(2\pi)^{D/2}\frac{\Gamma\left(\frac{d+s-l}{2}\right)}{\Gamma\left(\frac{l-d+s+D}{2}\right)}(-1)^{s+1}\frac{\Gamma(D-d)\Gamma\left(\frac{l+d+s-D}{2}\right)}{\Gamma(d-D/2+1)\Gamma\left(\frac{2D-l-d+s}{2}\right)}\\
&\quad\times\Big\{-\Big[\frac{1}{2}C_V(l+d+s-D) + \frac{g_R}{g_j}\frac{1}{2}(d+s-l)\\
&\qquad\times\frac{(l+d+s-D)(l+d-s-D)}{2(D-d-1)}\Big]C^l_{1\mu,\mu_1,\ldots\mu_s}(x_1x_2x_3)
\end{aligned}
$$

$$+\left[\frac{1}{2}C_V - \frac{g_R}{g_j}\left[(l+d-s-D) + \frac{(l-d+s+D-2)(2D-l-d+s-2)}{2(D-d-1)}\right]\right]$$
$$\times\, C^l_{2\mu,\mu_1,\ldots\mu_s}(x_1x_2x_3)\Bigg\}.$$

Now let us calculate a residue of this expressions at the points $l=d+s$. It is readily seen that the result satisfies the criterion (2.8) for all d and s if we set

$$g_R = \frac{C_V}{2(D-2)}g_j = \frac{C_V}{4}(2\pi)^{-D/2}\Gamma\left(\frac{D-2}{2}\right). \tag{2.10}$$

Thus the complete family of the secondary fields P_s which appears in the operator product expansion $j^a_\mu\varphi$ has well defined Green functions (2.9), provided that the operator product expansion $j^a_\mu(x)j^b_\nu(0)$ includes the field R^{ab}_j and the normalization of the Green function $\langle R^{ab}_j\varphi\varphi^+\rangle$ is given by Eq. (2.10).

3 A Family of Conformal Fields with Canonical Dimensions

Consider the operator product expansion $j^a_\mu R^{bc}$. It involves the tensor fields R_s which are analogous to the fields P_s. Apparently, the dimensions of the fields R_s are obtained by the substitution $d\to D-2$ in (1.2). We examine the fields which transform as vectors with respect to their inner index. So we have the family of the secondary fields

$$R_s = R^a_{\mu_1\ldots\mu_s}(x), \quad d_s = D-2+s \tag{3.1}$$

which arises in the operator expansion

$$j^a_\mu(x)R^{bc}(0) = [R^{ab}] + \sum_{s\geq 1}[R^q_s]. \tag{3.2}$$

The current j^a_μ belongs to this family: $j^a_\mu(x) = R^a_s|_{s=1}$.

The Green functions of the fields R^a_s are obtained from the expressions (2.6) after the substitution $d\to D-2$, while the existence criterion of these functions comes up when the same substitution is performed in Eq. (2.8). To analyze the Green functions it proves convenient to extract the traceless part $\hat R^{ab}$ of the field R^{ab}

$$R^{ab} = \hat R^{ab} + \frac{1}{N}\delta^{ab}R, \quad R = R^{aa},$$

and apply the Ward identities

$$\begin{aligned}
\partial^{x_4}_\nu\langle j^m_\nu(x_4)j^n_\mu(x_3)\hat R^{ab}(x_1)\hat R^{cd}(x_2)\rangle &= \delta(x_{34})f^{mnk}\langle j^k_\mu(x_3)\hat R^{ab}(x_1)\hat R^{cd}(x_2)\rangle \\
&+ \delta(x_{14})f^{mak}\langle j^n_\mu(x_3)\hat R^{kb}(x_1)\hat R^{cd}(x_2)\rangle + (a\leftrightarrow b) \\
&+ \delta(x_{24})f^{mck}\langle j^n_\mu(x_3)\hat R^{ab}(x_1)\hat R^{kd}(x_2)\rangle + (c\leftrightarrow d) \\
+\,\partial^{x_4}_\nu\delta(x_{34})\langle\hat R^{mn}(x_3)R^{ab}(x_1)\hat R^{cd}(x_2)\rangle &+ \frac{1}{N}\delta^{mn}\partial^{x_4}_\nu\delta(x_{34})\langle R(x_3)R^{ab}(x_1)\hat R^{cd}(x_2)\rangle
\end{aligned} \tag{3.3}$$

$$\partial^{x_4}_\nu\langle j^m_\nu(x_4)j^n_\mu(x_3)R^{ab}(x_1)R(x_2)\rangle = \partial^{x_4}_\nu\delta(x_{34})\langle R^{mn}(x_3)R^{ab}(x_1)R(x_2)\rangle. \tag{3.4}$$

Let us find out the Green function $\langle R^a_s \hat{R}^{cd} j^n_\mu \rangle$. It is determined by the equation analogous to Eq. (2.1):

$$\langle R^q_{\mu_1\dots\mu_s}(x_1)\hat{R}^{cd}(x_2)j^n_\mu(x_3)\rangle$$
$$= -\operatorname*{res}_{l=D-2+s}\int dx_4\,dx_5\,B^{q\ abm}_{\mu_1\dots\mu_s}(x_1x_5x_4)\partial^{x_4}_\nu\langle j^m_\nu(x_4)j^n_\mu(x_3)R^{ab}(x_5)\hat{R}^{cd}(x_2)\rangle, \quad (3.5)$$

where $B^{q\ abm}_{\mu_1\dots\mu_s}(x_1x_5x_4) = (\delta^{am}\delta^{bq}+\delta^{aq}\delta^{bm})\hat{B}^l_{\mu_1\dots\mu_s}(x_1x_5x_4)$, and $\hat{B}^l_{\mu_1\dots\mu_s}$ is obtained from the expressions (2.2) after the substitution $d \to D-2$.

To simplify the calculation, consider the case of the group $SU(2)$. We put $f^{abc}=\epsilon^{abc}$, $\delta^{aa}=N=3$, where ϵ^{abc} is the totally antisymmetric tensor. The Green functions in the r.h.s. of the Ward identity (3.3) read

$$\langle j^n_\mu(x_3)\hat{R}^{ab}(x_1)\hat{R}^{cd}(x_2)\rangle = h_j\left[f^{nac}\delta^{bd}+f^{nbc}\delta^{ad}+(c\leftrightarrow d)\right]\lambda^{x_3}_\mu(x_1x_2)\Delta_R(x_1x_2x_3),$$
$$\langle \hat{R}^{ab}(x_1)\hat{R}^{cd}(x_2)R(x_3)\rangle = h_1\left[\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}-\frac{2}{3}\delta^{ab}\delta^{cd}\right]\Delta_R(x_1x_2x_3),$$
$$\langle \hat{R}^{ab}(x_1)\hat{R}^{cd}(x_2)\hat{R}^{mn}(x_3)\rangle = h_2[\delta^{ac}\delta^{bm}\delta^{dn}+(a\leftrightarrow b)$$
$$+(c\leftrightarrow d)+(m\leftrightarrow n)-\text{traces}\Big]\Delta_R(x_1x_2x_3),$$

where h_1, h_2, h_3 are the constants, $\Delta_R(x_1x_2x_3) = (\frac{1}{8}x^2_{12}x^2_{13}x^2_{23})^{-\frac{D-2}{2}}$. Substitute these equations, as well as the identity (3.3), into Eq. (3.5). The expression of the residue part will be analogous to that in (2.4). Taking the integrals with the help of relations provided in [1] we find the following result: the r.h.s. of Eq. (3.5) satisfies the criterion of existence (see (2.6),(2.8) for $d\to D-2$) if

$$(D-2)\left(\frac{4}{3}h_1+\frac{14}{3}h_2\right)+3h_j=0. \quad (3.6)$$

When this requirement holds, the Green functions $\langle R_s\hat{R}j_\mu\rangle$ have the form[3]:

$$\langle R^q_{\mu_1\dots\mu_s}(x_1)\hat{R}^{ab}(x_2)j^n_\mu(x_3)\rangle \sim (\delta^{aq}\delta^{bn}+\delta^{an}\delta^{bq}-\frac{1}{N}\delta^{ab}\delta^{qn})$$
$$\times (x^2_{23})^{-\frac{D-2}{2}}\overleftrightarrow{\partial}^{x_3}_\mu\left[(x^2_{13})^{-\frac{D-2}{2}}\lambda^{x_1}_{\mu_1\dots\mu_s}(x_3x_2)\right](x^2_{12})^{-\frac{D-2}{2}}+\textit{quasilocal terms}. \quad (3.7)$$

Using the Ward identity (3.4) one can show that the Green functions $\langle R^q_{\mu_1\dots\mu_s}Rj^a_\mu\rangle$ where $R=R^{aa}$ contain only quasilocal terms which are given by the expression (2.7) after the substitution $d\to D-2$.

Consider the higher Green functions of the fields R_s. They are determined from the equation

$$\langle R^a_s(x)\varphi(x_1)\dots\varphi^+(x_{2n})\rangle = -(\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})$$
$$\times \operatorname*{res}_{l=D-2+s}\int dy_1\,dy_2\,\hat{B}^l_{\mu_1\dots\mu_s}(xy_1y_2)\partial^{y_2}_\nu\langle j^b_\nu(y_2)R^{cd}(y_1)\varphi(x_1)\dots\varphi^+(x_{2n})\rangle. \quad (3.8)$$

The method used to calculate the r.h.s. is described in detail in [1], see also [2].

[3]For even D these expressions demand more precise definition. This problem will be addressed in the other article.

Consider this equation for $s=1$. As mentioned above, the field R^a_s for $s=1$ coincides with the current j^a_μ. Using the methods developed in [1] one can show that for $s=1$ Eq. (3.8) leads to the following relation (taking into account that $R^a_s|_{s=1}=j_\mu$):

$$\langle j^a_\mu(x)\varphi(x_1)\ldots\varphi^+(x_{2n})\rangle = \gamma\sum_{k=1}^{2n} n\frac{(x-x_k)_\mu}{(x-x_k)^2}t^b_{x_k}\langle R^{ab}(x)\varphi(x_1)\ldots\varphi^+(x_{2n})\rangle. \qquad (3.9)$$

The constant γ is calculated from the same equation for $n=1$. Taking into account (2.10), we have: $\gamma = g_j/g_R = \frac{C_V}{2(D-2)}$. The relation (2.9) and the Ward identity for the Green functions $\langle j_\mu\varphi\ldots\rangle$ lead to the following equation

$$\gamma\partial^x_\mu\sum_{k=1}^{2n}\frac{(x-x_k)_\mu}{(x-x_k)^2}t^b_{x_k}\langle R^{ab}(x)\varphi(x_1)\ldots\varphi^+(x_{2n})\rangle = -\sum_{k=1}^{2n}\delta(x-x_k)t^a_{x_k}\langle\varphi(x_1)\ldots\varphi^+(x_{2n})\rangle \qquad (3.10)$$

4 Secondary Fields Generated by the Energy-Momentum Tensor

The results analogous to those described above may be obtained for the secondary fields generated by the energy-momentum tensor. Let us consider them qualitatively.

The requirement for existence of the fields P_s coincides with the condition that the Green functions $\langle P^T_s\varphi T_{\mu\nu}\rangle$ are well defined. For $D\geq 3$, the general conformally invariant expression of the functions $\langle P^T_s\varphi T_{\mu\nu}\rangle$ has the form:

$$\langle P^T_s(x_1)\varphi(x_2)T_{\mu\nu}(x_3)\rangle = \lim_{l=d+s}\left[\sum_{i=1}^{3}A_i(l,s)C^l_{i,s}(x_1x_2x_3)\right], \qquad (4.1)$$

where $C^l_{i,s}(x_1x_2x_3)$ are the basic conformal functions [1,2] and A_i are arbitrary coefficients. All the functions $C^l_{i,s}$ are poorly defined for $l=d+s$. However, the limit of the sum in (4.1) is a well defined function provided that [1,2]

$$\lim_{l=d+s}\frac{A_1(l,s)}{A_2(l,s)} = D(D-2), \quad \lim_{l=d+s}\frac{A_2(l,s)}{A_3(l,s)} = D. \qquad (4.2)$$

The Green functions $\langle P^T_s\varphi T_{\mu\nu}\rangle$ must be found from the equation analogous to Eq. (2.1). Its r.h.s. includes the quantity $\partial_\mu\langle T_{\mu\nu}T_{\rho\sigma}\varphi\varphi\rangle$ determined by an anomalous Ward identity. This Ward identity was derived in [1] (see also [2]). The calculations are carried out in the same manner as in the case of the current. As the result, the criterion (4.2) leads to certain conditions which impose the relations between the normalization of the Green functions $\langle R_T\varphi\varphi\rangle$ (where R_T is the anomalous contribution in the expansion $TT=[R_T]+\ldots$) and of the other free parameters [1,2] entering the anomalous Ward identity for $\langle T_{\mu\nu}T_{\rho\sigma}\varphi\varphi\rangle$.

The operator product expansion

$$T_{\mu\nu}(x)R_T(0) = \sum_s [R^T_s]$$

where $R^T_s = R_{\mu_1\ldots\mu_s}$ are the tensor fields of canonical dimensions $D-2+s$, is handled analogously. The condition of their existence is derived from Eqs. (4.1) and (4.2) after

the substitution $d \to D-2$. Identically to the case of the current, this condition fixes up the normalization of the Green function $\langle R_T R_T R_T \rangle$ which enters the anomalous Ward identity for $\langle T_{\mu\nu} T_{\rho\sigma} R_T R_T \rangle$. The field $R_T(x)$ is the fundamental field of the family of fields R_s^T. The energy-momentum tensor belongs to this family: $T_{\mu\nu}(x) = R_s^T|_{s=2}$. Using this, one can express the Green functions $\langle T_{\mu\nu}\varphi \ldots \varphi \rangle$ in terms of the functions $\langle R\varphi \ldots \varphi \rangle$, and to find the equation for the functions $\langle R\varphi \ldots \varphi \rangle$, analogously to the case of Eqs. (3.9),(3.10). All necessary calculations will be presented elsewhere.

Acknowledgments

The work was supported in part by grants RFBR No. 05-02-17654, LSS No. 1578.2003-2 and INTAS No. 03-51-6346.

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